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Computational Analysis of the Governing Equations of the Topographic Waves in a Homogenous Ocean

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***Abstract*– Various external forces influence water movements in a homogenous sea. The topographic equations governing the topographic waves in a homogenous sea have been elaborated in this study. The computational analysis pertinent to delineating the topographic waves in the homogenous sea has been emphasized. The bottom slope and friction factor determines the variation trend of the resultant ocean velocity and the surface ocean depth and generated results for this have been presented in this study.**

***Index Terms*– Topographic Waves, Geotropic, Bottom Slope, Bottom Friction Factor and Homogenous Ocean**

1. INTRODUCTION

THE ocean surface is an example of a complex wave motion formed by the action of wind. The displacement of a fluid particle from equilibrium position and action of a restoring (gravitational) force on the particle produces a wave like motion in the ocean called an internal wave.

The motion of ocean water is strongly influenced by the spatial variation in homogeneity of the wind field over the ocean surface and the topography of the ocean bottom. Topographic waves are modeled using a primitive-equation ocean model.

Various external forces influence water movements in a homogenous sea. These comprise major forces that maintain the ocean currents including air currents, the changes in atmosphere pressure at the surface of the sea and the periodic tide-generating astronomic forces. The changes in atmospheric pressure are transmitted through the entire mass of water down to the ocean bottom and this give rise to horizontal pressure differences and the formation gradient currents. The air currents result to two fold effects consisting of the tangential force of the ocean(wind stress) which produces a surface current transmitted by the effect of viscosity(turbulence) to the water layers waves also constitute water movements in the direction of the wind.

Internal forces arise from the vertical and horizontal disturbances of mass within the ocean. These differences in the mass distribution both in the horizontal and vertical directions are the consequences of changes in the heat content (temperature) and in the salinity.

A. Equation of Motion

The product of mass and acceleration equals the vector sum of forces as asserted by Newton's second law of motion. This statement is invariably called the equation of motion.

The important forces which drive the large-scale motion are the force of gravity, the Coriolis force, pressure gradient force and frictional forces. The centrifugal force of earth's rotation is usually included in gravity. The three dimensional acceleration of a particle is described by the vector equation of motion, which contains the following terms:

Particle acceleration = Coriolis term + Presure gradient term + Gravity trems + frictional term and expressed as;

$$\frac{dc}{dt} = (-2u \times c) + \left(\frac{-1}{\rho}\nabla P\right) + g + F \quad \ldots\ldots\ldots\ldots (1.0)$$

where dc/dt is the acceleration of a unit mass due to accumulated effects per unit mass of the Coriolis force $-2u \times c$, the pressure gradient force $-1/\rho\nabla P$, the force of gravity g and F, the generalized force due to frictional effects. The above equation can be written as;

$$\frac{du}{dt} = (2\Omega sin\Phi)v - \frac{1}{\rho}\frac{\partial P}{\partial x} + F_x \quad \ldots\ldots (1.1)$$

$$\frac{du}{dt} = (2\Omega sin\Phi)u - \frac{1}{\rho}\frac{\partial P}{\partial x} + F_y \quad \ldots\ldots (1.2)$$

$$\frac{dw}{dt} = +g - \frac{1}{\rho}\frac{\partial P}{\partial z} + F_z \quad \ldots\ldots (1.3)$$

II. METHODOLOGY

The governing equations of the topographic waves in a homogenous sea have been vividly delineated in this investigation. The computational analysis of the governing equations was treated by deriving the pertinent analytic expressions.

The salient features of the pertinent equations were unveiled and the input parameters requisite for the computational task were stated.

III. DISCUSSION

Many types of waves involving different physical factors exist in the ocean. An analogy could be made to an elementary spring-mass system, thus all waves must be associated with some kind of restoring force equivalent to an elementary spring-mass system or simple pendulum, and as a result it is convenient to make a crude classification of ocean waves.

A. Topographic Waves and Dynamics of Ocean Bottom

Small bottom irregularities can turn an otherwise steady geostrophic flow into slow moving waves. The dynamics of an ocean with bottom slope is elaborated here.

For simplicity an homogenous ocean is considered in a domain with periodic boundaries in y and a weak uniform bottom slope in the x direction as delineated by pertinent equations below:

Emphasizing the vertically integrated continuity equation:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)\frac{\partial}{\partial y}(hv)=0 \quad \text{..........................} \; (1.4)$$

By substituting $h(x,y,z)=H_0-\alpha x+\eta(x,y,t)$

into equation (1.4) above gives;

$$\frac{\partial h}{\partial y}=\frac{\partial \eta}{\partial y} \quad \text{..........................} \; (1.5)$$

$$\frac{\partial}{\partial x}(hu)=u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x} \quad \text{..........................} (1.6)$$

$$\frac{\partial}{\partial x}(hv)=v\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x} \quad \text{..........................} (1.7)$$

$$\frac{\partial h}{\partial x}=-\alpha+\frac{\partial \eta}{\partial x} \quad \text{..........................} (1.8)$$

$$\frac{\partial h}{\partial t}=\frac{\partial \eta}{\partial t} \quad \text{..........................} (1.9)$$

By substituting equation (1.5) into equation (1.4) yields;

$$\frac{\partial \eta}{\partial t}+\left(u\frac{\partial h}{\partial x}+v\frac{\partial \eta}{\partial y}\right)+\left(H_0-\alpha x+\eta\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\alpha_0 u=0 \quad \text{..........} (2.0)$$

From linear theory and requirement of a gentle slope, the continuity equation is written as follows:

$$\frac{\partial \eta}{\partial t}+(H_0)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\alpha_0 u=0 \quad \text{..........................} (2.1)$$

The corresponding linear vertically integrated momentum equations are:

$$\frac{\partial u}{\partial t}-fv=-g\frac{\partial \eta}{\partial x} \quad \text{..........................} (2.2)$$

$$\frac{\partial v}{\partial t}-fu=-g\frac{\partial \eta}{\partial x} \quad \text{..........................} (2.3)$$

The extra term $\alpha_0 u$ in the continuity equation, related to the bottom slope will allow the existence of slow waves similar to the planetary waves due to the variation of th Coriolis parameter. This system contains both small and large terms. The large ones (terms including f,g and H_0) comprise the otherwise steady geostrophic dynamics. In the presence of the small term $\alpha_0 u$, the time derivatives come into play, but are still expected to be small. Thus based on this smallness, we can take as a small approximation, the geostrophic balance:

$$u=\left(\frac{g}{f}\right)\frac{\partial \eta}{\partial y}, \quad v=\left(\frac{g}{f}\right)\frac{\partial \eta}{\partial x} \quad \text{..........................} (2.4)$$

By substituting equations (2.4) in the small time derivatives of equations (2.2) and (2.3), we obtain;

$$\frac{\partial u}{\partial t}=-\left(\frac{g}{f}\right)\frac{\partial^2 \eta}{\partial y\partial t} \quad \text{..........................} (2.5)$$

$$\frac{\partial v}{\partial t}=-\left(\frac{g}{f}\right)\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (2.6)$$

From equation (2.3);

$$fu=-g\frac{\partial \eta}{\partial y}-\frac{\partial v}{\partial t} \quad \text{..........................} (2.7)$$

$$u=-\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{1}{f}\frac{\partial v}{\partial t} \quad \text{..........................} (2.8)$$

$$u=-\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{1}{f}\left(\frac{g}{f}\right)\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (2.9)$$

$$u=-\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (3.0)$$

Similarly for v;

$$fv=\frac{\partial u}{\partial t}+\frac{\partial \eta}{\partial x} \quad \text{..........................} (3.1)$$

$$v=\frac{1}{f}\frac{\partial u}{\partial t}+\frac{g}{f}\frac{\partial \eta}{\partial x} \quad \text{..........................} (3.2)$$

$$v=\frac{1}{f}\left(-\frac{g}{f}\right)\frac{\partial^2 \eta}{\partial x\partial t}+\frac{g}{f}\frac{\partial \eta}{\partial x} \quad \text{..........................} (3.3)$$

$$v=\frac{g}{f}\frac{\partial \eta}{\partial x}-\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (3.4)$$

By replacement of the component in the continuity equation (2.1) yields a single equation for ⍰ as follows;

$$u=-\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (3.5)$$

$$u=-\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (3.6)$$

$$\frac{\partial u}{\partial x}=-\frac{g}{f}\frac{\partial^2 \eta}{\partial x\partial y}-\frac{g}{f^2}\left(\frac{\partial^2 \eta}{\partial^2 x}\right) \quad \text{..........................} (3.7)$$

$$v=\frac{g}{f}\frac{\partial \eta}{\partial y}-\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x\partial t} \quad \text{..........................} (3.8)$$

$$\frac{\partial v}{\partial y}=\frac{g}{f}\frac{\partial^2 \eta}{\partial x\partial y}-\frac{g}{f}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x\partial t}\right) \quad \text{..........................} (3.9)$$

By substituting equations (3.7), (3.4) and (2.9) into equation (2.1) yields;

$$\frac{\partial \eta}{\partial t}-R^2\frac{\partial}{\partial t}\nabla^2\eta+\frac{\alpha_0 g}{f}\frac{\partial \eta}{\partial y} \quad \text{..........................} (3.91)$$

where $R=\frac{\sqrt{gH_0}}{f}$. This is the Rossby radius.

The solution of equation (3.91) gives;

$$\eta=Acos(lx+my-\omega t) \quad \text{..........................} (3.92)$$

$$\frac{\partial \eta}{\partial x}=-Alsin(lx+my-\omega t) \quad \text{..........................} (3.93)$$

$$\frac{\partial \eta}{\partial y}=-Amsin(lx+my-\omega t) \quad \text{..........................} (3.94)$$

$$\frac{\partial \eta}{\partial t}=A\omega sin(lx+my-\omega t) \quad \text{..........................} (3.96)$$

$$\frac{\partial^2 \eta}{\partial x^2}=-Al^2cos(lx+my-\omega t) \quad \text{..........................} (3.97)$$

$$\frac{\partial^2 \eta}{\partial y^2}=-Am^2cos(lx+my-\omega t) \quad \text{..........................} (3.98)$$

Substitution of equations (3.93) – (3.98) into equation (3.91) above gives the dispersion relation expressed as;

$$\omega = \frac{\alpha_0\ g}{f}\ \frac{m}{(1+R^2(l^2+m^2))} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.99)$$

These waves exist on their own due to the existence of the bottom slope α_0, hence they are called topographic waves. Without the presence of the bottom slope α_0, the flow would be steady and geostrophic.

IV. COMPUTATIONAL ANALYSIS

Having delineated the governing equations of the topographic waves in a homogenous ocean previously, the computational procedure required to in solving the differential equations are listed subsequently.

The linear, vertical integrated momentum equations are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.0)$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

Geostrophic balance requires; $u \approx -(\frac{g}{f})\frac{\partial \eta}{\partial y}$.

These equations would be solved numerically by adopting a numerical scheme (leap frog) in time placed in a 2-D staggered grid (Arakawa Grid) whose complete scheme is listed as follows:

$$\frac{u_{jk}^{n+1}-u_{jk}^{n-1}}{2\Delta t} = f\left[\frac{v_{j,k+1}^{n}+v_{j+1,k+1}^{n}+v_{jk}^{n}+v_{j+1,k}^{n}}{4}\right]\text{-g}\left[\frac{\eta_{j+1,k}-\eta_{j-1,k}}{2\Delta x}\right]$$

………..(4.2)

for the u equation centered at u_{jk} .

$$\frac{v_{jk}^{n+1}-v_{jk}^{n-1}}{2\Delta t} = f\left[\frac{\eta_{j-1,k}^{n}+u_{jk}^{n}+u_{j-1,k-1}^{n}+u_{j,k-1}^{n}}{4}\right]\text{-g}\left[\frac{\eta_{j,k}-\eta_{j,k-1}}{2\Delta y}\right]$$

……….(4.3)

for the v equation centered at v_{jk} .

For the ▯ equation centered at $▯_{jk}$, the boundary conditions are periodic at y=0 and y=Ly, and no slip condition at the walls x=0 and x=Ly. This subsequent expression is implemented by updating at every time step the t tangential velocities inside the boundaries to a value equal to the negative of the velocity at the point immediately outside the boundary.

The shuffling of the time levels is done by changing the indices and not the variables it solves, i.e;

$▯_{save}$=▯+1, ▯+1=▯+2, ▯+2=$▯_{save}$ respectively. A forcing term (wind) is included in the program to start the currents; the wind is shut down after one day.

- *Input:*

The input parameters required to run the program in the model are as follows:

$\Delta t = 305$

$K_{max} = 40$

$J_{max} = 20$

$\Delta x/2 = \Delta y/2 = 5$km

Ly = 400km, Lx = 200km

$H_0 = 100$m

$T^x = 0$

$$\tau^y = \begin{cases} \tau \sin\left(\frac{2\pi y}{N_y^2}\right) & T \le 1\ day \\ 0 & T > 1\ day \end{cases}$$

α=0

The values of the bottom slope α and bottom friction factor can be obtained, the results for α = 0, r= 0 and n = 180 are listed in the Table 4.

Table 4:

C (m/s)	**▯ (m)**
3.00	2.40
4.86	4.80
8.96	9.60
10.61	12.00
14.82	17.00
6.63	2.40
7.52	7.20
11.67	14.00
6.27	4.80
6.59	7.20

C represents the resultant ocean wave velocity; ▯ is the surface ocean depth.

V. CONCLUSION

The resultant ocean velocities and corresponding surface ocean depths are presented in table 4.0 from this investigation. The values of α and r are zero respectively in this investigation, which determine the variation trend of the ocean wave velocity with the surface ocean depth.

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